What WFFS Are

and

How to Find Them

Notes for Allen/Hand, *Logic Primer* 1.2

Vocabulary: The symbols of our language are:

- **Sentence Letters:** $A, B, C, ..., A_1, B_1, C_1, ...$
- **Connectives:** $\sim, \&, \vee, \rightarrow, \leftrightarrow$
- **Parentheses:** (and)

Expression: Any string of the symbols of our language

Well-Formed Formula: An expression built according to the rules for well-formed formulas.

There are seven rules, but it's easier to think of them as divided into three groups:

Rule 1: A statement letter alone is a wff

The wffs defined by this rule are all exactly **one character long**:

$$A$$

$$B$$

$$Z_4$$
$$C_0$$

This is the only rule that a one-character wff can satisfy.

Rule 2: If ϕ is a wff, then $\sim \phi$ is a wff

Every wff defined by this rule **begins with the symbol** $\sim$:

$$\sim A$$

$$\sim C$$

$$\sim\sim E$$

This is the only rule that can introduce a '$\sim$' into a wff

If an expression **begins with** $\sim$, then it is a wff if and only if what remains after you remove the initial $\sim$ is a wff.

Rules 3-6: If ϕ and ψ are both wffs, then each of these is a wff

Rule 3: $(\phi \& \psi)$

Rule 4: $(\phi \vee \psi)$

Rule 5: $(\phi \rightarrow \psi)$

Rule 6: $(\phi \leftrightarrow \psi)$

Notice that every wff defined by any of these rules **begins with '(' and ends with ')'.** These are the only rules that introduce parentheses into wffs. Moreover, each rule introduces a matched pair of left and right parentheses:

- If a wff begins with '(', then it must end with ')'.

- If a wff begins with '(', then it must match one of Rules 3-6.

- Every '(' must be matched by one and only one ')'.

Let's combine these points into the beginnings of a method.

To tell whether an expression is a wff:

1. If it begins with '$\sim$', then it must match Rule 2. Remove the '$\sim$' and see whether the remainder of the expression is a wff.

2. If it begins with '(', then it must match one of Rules 3–6. Determine whether it matches one of these rules.

3. If it begins with anything else, then it must match Rule 1. In that case, the only possibility is that it is a sentence letter standing alone.

Note that the only things a wff can begin with are $\sim$, (, and a sentence letter.

Step 2 here needs to be filled in. Since parentheses always occur in pairs, we can start by looking for the mate of the initial '('. Starting at the beginning of the formula, move to the right one character at a time and count as follows:

- 2a. If the next character is '(', add 1 to the count.
- 2b. If the next character is ')', subtract 1 from the count.
- 2c. If the next character is anything else, leave the count unchanged.
- 2d. Stop when *either* the count is 0 *or* you come to the end of the expression.

When you come to a stop, there will be only two possibilities:

1. The count is 0 and you have stopped at a ')'. You have found the mate of the '(' that you started with.

2. The count is not 0 and you have run out of characters. **The expression is not a wff.**

An illustration of this appears on the next slide.

$$((A \rightarrow B) \vee \sim P)$$

$$\overbrace{(}^{count=1} \; (A \rightarrow B) \vee \sim P)$$

$$(\; \overbrace{(}^{count=2} \; A \rightarrow B) \vee \sim P)$$

$$((\; \overbrace{A}^{count=2} \; \rightarrow B) \vee \sim P)$$

$$((A \; \overbrace{\rightarrow}^{count=2} \; B) \vee \sim P)$$

$$((A \rightarrow \; \overbrace{B}^{count=2} \;) \vee \sim P)$$

$$((A \rightarrow B \; \overbrace{)}^{count=1} \; \vee \sim P)$$

$$((A \rightarrow B) \; \overbrace{\vee}^{count=1} \; \sim P)$$

$$((A \rightarrow B) \vee \; \overbrace{\sim}^{count=1} \; P)$$

$$((A \rightarrow B) \vee \sim \; \overbrace{P}^{count=1} \;)$$

$$((A \rightarrow B) \sim P \; \overbrace{)}^{count=0}$$

We got to the end, and the count was zero. Therefore, we know that the first '(' and the last ')' of $(A \rightarrow B) \vee \sim P$ match up.

We need to add a little more to Step 2. Rules 3–6 all give us wffs of the form:

$$(\phi \ connective \ \psi)$$

So, once we have found the matching outside parentheses, we remove them and see whether what is left is of the form:

$$\phi \ connective \ \psi$$

Here is what we should look for, then:

1. Starting at the left, look for a wff at the beginning.

2. The *next symbol after* that wff must be &, $\vee$, $\rightarrow$, or $\leftrightarrow$.

3. The rest of the expression after that symbol must be a wff

Let's try this with our example $((A \rightarrow B) \vee \sim P)$. When we take off the outside parentheses, we have $(A \rightarrow B) \vee \sim P$. The rest of the steps are on the next slide.

$$(A \rightarrow B) \vee \sim P$$

$$\overbrace{(}^{count=1} A \rightarrow B) \vee \sim P$$

$$(\overbrace{A}^{count=1} \rightarrow B) \vee \sim P$$

$$(A \overbrace{\rightarrow}^{count=1} B) \vee \sim P$$

$$(A \rightarrow \overbrace{B}^{count=1}) \vee \sim P$$

$$(A \rightarrow B \overbrace{)}^{count=0} \vee \sim P$$

We stop at the count of zero, and now we have found the first expression:

$$(A \rightarrow B)$$

Is this a wff? We proceed as before, taking off the outside parentheses:

$$A \rightarrow B$$

Let's start at the beginning as before. Here, the first symbol A is neither a '(' nor a '$\sim$'. So, we must use **Rule 1**: it must be a sentence letter standing alone.

Since we've found a wff, the next question is, "Is the next symbol one of &, $\vee$, $\rightarrow$, $\leftrightarrow$?" Here is what we find:

$$\overbrace{A}^{wff} \overbrace{\rightarrow}^{next symbol} B$$

The next symbol is indeed one of the four used in Rules 3–6, and so all that remains is to see whether the rest of the formula is a wff:

$$\overbrace{A}^{wff}\ \overbrace{\rightarrow}^{nextsymbol}\ \overbrace{B}^{wff?}$$

B is a wff by Rule 1, so we've satisfied all of the conditions. Summarizing where we are, then, we have found a wff that matches Rule 4:

$$(A \rightarrow B)$$

Going back to our larger formula, this is the wff on the left side. Let's call it ϕ. Our next step is to see whether the very next symbol after it is a connective of the right sort:

$$\overbrace{(A \rightarrow B)}^{\phi}\ \overbrace{\vee}^{nextsymbol}\ \sim P$$

It is: it's the wedge, used in Rule 2. Our expression will satisfy that rule if everything after the wedge is a wff. Let's call that ψ:

$$\overbrace{(A \rightarrow B)}^{\phi} \vee \overbrace{\sim P}^{\psi}$$

So, our final question is: is ψ a wff? Since it begins with $\sim$, we need to use Rule 2.

$$\psi = \sim P$$

Since this begins with $\sim$, it can be a wff only if it matches Rule 2. To check this, we remove the $\sim$ and see if the result is a wff:

$$P$$

But it is, so we're done.

Now let's put that back together into our original wff. We start with two wffs, the ones we just found:

$$(A \rightarrow B) \qquad\qquad \sim P$$

We then combine them using Rule 4:

$$((A \rightarrow B) \vee \sim P)$$

So, the entire expression corresponds to the form given in Rule 4:

$$(\overbrace{(A \rightarrow B)}^{\phi} \vee \overbrace{\sim P}^{\psi})$$

The whole method is summarized in the next slide.

1. If it begins with '$\sim$', then it must match Rule 2. Remove the '$\sim$' and see whether the remainder of the expression is a wff.

2. If it begins with '(',

   (a) Count parentheses from left to right as follows:

      i. If the next character is '(', add 1 to the count.

      ii. If the next character is ')', subtract 1 from the count.

      iii. If the next character is anything else, skip it.

      iv. Stop when *either* the count is 0 *or* you come to the end of the expression.

   (b) If you reach 0 before the end of the expression or you reach the end of the expression and the count is not 0, then stop: it's not a wff.

   (c) Otherwise, remove the outside parentheses and:

      i. Find the first wff from the left.

      ii. The next symbol must be &, $\vee$, $\rightarrow$, or $\leftrightarrow$. If not, stop: it's not a wff.

      iii. The rest of the expression must be a wff. If not, stop: it's not a wff.

3. If it begins with anything else, then it must be a sentence letter and match Rule 1. If not, stop: it's not a wff.

Recursive Rules

The rules for **constructing** wffs have an important property. They can be applied to the wffs that result from applying them. To use a technical term for this, they are **recursive**.

Since the result of applying a rule to wffs is a new wff, and since this process can be continued indefinitely, the rules for wffs define an **infinite** set of wffs.